Solution tentamen Signal Analysis 31/1/06

1) a. $\mathcal{F}\{f(t)\} = \int_0^\infty e^{-t/\tau}\, e^{-i2\pi f t}\, dt = \left[\frac{1}{-\frac{1}{\tau} - i2\pi f} e^{-t\left(\frac{1}{\tau} + i2\pi f\right)}\right]_0^\infty = \frac{1}{\frac{1}{\tau} + i2\pi f} = \frac{\tau}{1 + i2\pi f\tau}$

$\mathcal{F}\{g(t)\} = \int_{-\infty}^0 e^{t/\tau}\, e^{-i2\pi f t}\, dt = \frac{\tau}{1 - i2\pi f\tau}$

b. $\mathcal{F}\{\varphi(-t)\} = \int_{-\infty}^{\infty} \varphi(-t)\, e^{-i2\pi f t}\, dt \overset{t=-t'}{=} \int_{\infty}^{-\infty} \varphi(t')\, e^{i2\pi f t'} \cdot -dt' = \int_{-\infty}^{\infty} \left[\varphi^*(t')\, e^{-i2\pi f t'}\right]^* dt' =$

$= \left[\mathcal{F}\{\varphi^*(t)\}\right]^*$

$\mathcal{F}\{g(t)\} = \mathcal{F}\{f(-t)\} = \left[\mathcal{F}\{f^*(t)\}\right]^* \overset{f(t)\text{ is real}}{=} \left[\mathcal{F}\{f(t)\}\right]^* = \left[\frac{\tau}{1 + i2\pi f\tau}\right]^* \overset{!}{=} \frac{\tau}{1 - i2\pi f\tau}$

c. $\mathcal{F}\{f\}\, \mathcal{F}\{g\} = \frac{\tau}{1 + i2\pi f\tau}\, \frac{\tau}{1 - i2\pi f\tau} = \frac{\tau^2}{1 + (2\pi f\tau)^2}$

$\frac{\tau}{2}\left[\mathcal{F}\{f\} + \mathcal{F}\{g\}\right] = \frac{\tau}{2}\left[\frac{\tau}{1 + i2\pi f\tau} + \frac{\tau}{1 - i2\pi f\tau}\right] = \frac{\tau}{2}\left[\frac{2\tau}{1 + (2\pi f\tau)^2}\right] \overset{!}{=} \mathcal{F}\{f\}\, \mathcal{F}\{g\}$

$\mathcal{F}^{-1}\left[\mathcal{F}\{f\}\, \mathcal{F}\{g\}\right] = f * g = \mathcal{F}^{-1}\left[\frac{\tau}{2}\left(\mathcal{F}\{f\} + \mathcal{F}\{g\}\right)\right] = \frac{\tau}{2}\left[\mathcal{F}^{-1}\mathcal{F}\{f\} +\right.$

$\left. + \mathcal{F}^{-1}\mathcal{F}\{g\}\right] \overset{!}{=} \frac{\tau}{2}\left[f + g\right]$

d. $f * g = \int_{-\infty}^{\infty} \underbrace{e^{-t'/\tau}}_{f,\ t' \geq 0}\ \underbrace{e^{(t-t')/\tau}}_{\substack{g,\ t-t' \leq 0 \\ \Rightarrow t' \geq t}}\, dt'$

Case 1: $t \leq 0 \Rightarrow \int_0^\infty e^{-t'/\tau}\, e^{(t-t')/\tau}\, dt' = e^{t/\tau}\left[-\frac{\tau}{2} e^{-2t'/\tau}\right]_0^\infty = \frac{\tau}{2} e^{t/\tau}$

Case 2: $t \geq 0 \Rightarrow \int_t^\infty e^{-t'/\tau}\, e^{(t-t')/\tau}\, dt' = e^{t/\tau}\left[-\frac{\tau}{2} e^{-2t'/\tau}\right]_t^\infty = e^{t/\tau}\, \frac{\tau}{2}\, e^{-2t/\tau} =$

$= \frac{\tau}{2} e^{-t/\tau}$

$\Rightarrow f * g = \frac{\tau}{2} e^{-|t|/\tau} \overset{!}{=} \frac{\tau}{2}\left(f(t) + g(t)\right)$

2) a. $g(t) = \cos(2\pi f_1 t) \cdot \frac{1}{2}\left[\cos(2\pi(f_2 - f_3)t) + \cos(2\pi(f_2 + f_3)t)\right] =$

$= \frac{1}{4}\left[\cos(2\pi(f_1 - f_2 + f_3)t) + \cos(2\pi(f_1 + f_2 - f_3)t) +\right.$

$\left. + \cos(2\pi(f_1 - f_2 - f_3)t) + \cos(2\pi(f_1 + f_2 + f_3)t)\right]$

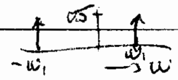
(2a continued) Frequency components: $f_1 - f_2 + f_3 = 150$ Hz

$f_1 + f_2 - f_3 = 50$ Hz

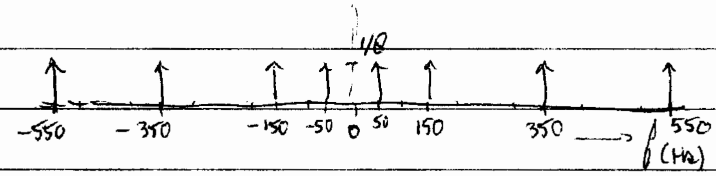
$f_1 - f_2 - f_3 = -350$ Hz (or 350 Hz)

$f_1 + f_2 + f_3 = 550$ Hz

$\cos \omega_1 t = \dfrac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2}$

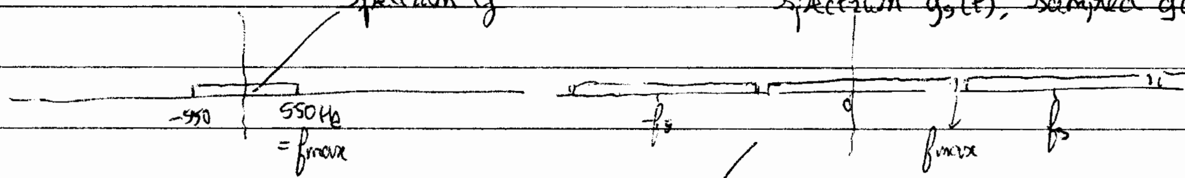


$\Rightarrow g(t)$



Amplitude: $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

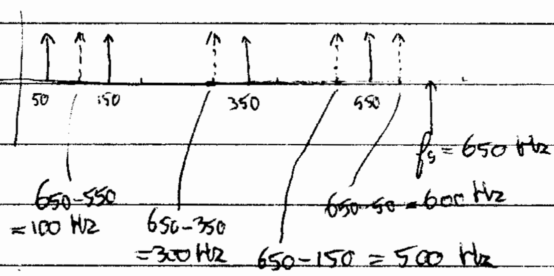
b. Nyquist frequency twice maximum frequency in $g(t)$ $\Rightarrow$ 1100 Hz



sampling produces multiple copies of spectrum, at $(\cdots -2f_s, -f_s, 0, f_s, 2f_s \cdots)$

Original spectrum can be retrieved if $f_s \geq 2 f_{max}$

c.



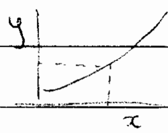
$\uparrow$ = original spectrum, centered at 0

$\hat{\uparrow}$ = aliased spectrum, centered at 650 Hz

$\Rightarrow$ frequencies present in $g_2(t)$: 50, 100, 150, 300, 350, 500, 550, and 600 Hz

3) a. $F_X(x) = \int_{-\infty}^{x} p_X(x')\,dx'$ $\qquad p_X(x) = \dfrac{dF_X(x)}{dx}$

b.

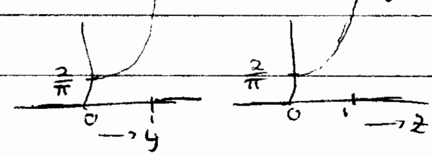


$F_Y(y) = P\{Y \leq y\} = P\{X \leq x\} = F_X(x)$

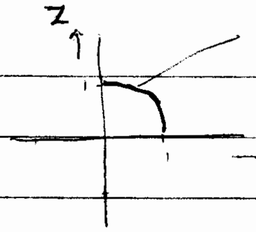
$\Rightarrow p_Y(y) = \dfrac{dF_Y(y)}{dy} = \dfrac{dF_X(x)}{dx}\dfrac{dx}{dy} = p_X(x)\dfrac{dx}{dy}$

c. $x = \arcsin y \Rightarrow p_Y(y) = p_X(x)\left|\dfrac{d \arcsin y}{dy}\right| = \dfrac{2}{\pi\sqrt{1-y^2}}$ $\qquad x \in (0, \frac{\pi}{2}) \Rightarrow y \in (0,1)$

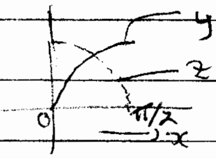
$p_Z(z) = p_X(x)\left|\dfrac{d \arccos z}{dz}\right| = \dfrac{2}{\pi\sqrt{1-z^2}}$ $\qquad z \in (0,1)$



3d



nonzero on this line, zero elsewhere; line is $\frac{1}{4}$ circle segment, because



$y^2 + z^2 = 1$,
and $0 \le y \le 1$
$0 \le z \le 1$

Definition independence: $p_{Y,Z}(y,z) = p_Y(y) \cdot p_Z(z)$. This is clearly not true here, because the circle segment is not equal to $\frac{2}{\pi\sqrt{1-y^2}} \cdot \frac{2}{\pi\sqrt{1-z^2}}$ (nonzero for any combination of y and z on the unit square bounded by (0,0), (0,1), (1,0), (1,1))

4) a. $P_x(f) = \mathcal{F}\{R_x(\tau)\} = \int_{-\infty}^{\infty} e^{-|\tau|} e^{-i2\pi f\tau} d\tau =$

$= \int_{-\infty}^{0} e^{\tau} e^{-i2\pi f\tau} d\tau + \int_{0}^{\infty} e^{-\tau} e^{-i2\pi f\tau} d\tau = \frac{1}{1-i2\pi f} + \frac{1}{1+i2\pi f} = \frac{2}{1+(2\pi f)^2}$

b. $R_y(\tau) = \mathcal{F}^{-1}\{0.1\} = 0.1\,\delta(\tau)$

x and y independent, stationary, and y zero-mean

$\Rightarrow P_s = P_x + P_y = \frac{2}{1+(2\pi f)^2} + 0.1$

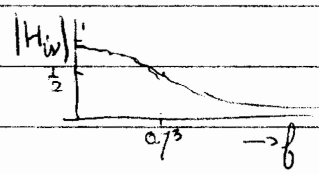
$R_s = R_x + R_y = e^{-|\tau|} + 0.1\,\delta(\tau)$

c. $H(f) = \mathcal{F}\{h(t)\} = \int_0^T \frac{1}{T} e^{-2\pi i f t} dt = \frac{1}{T}\frac{1}{-2\pi i f}\left[e^{-2\pi i f t}\right]_0^T =$

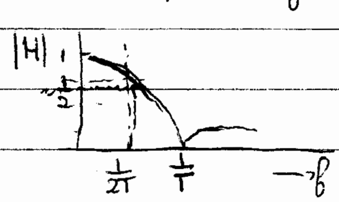
$= \frac{1}{T}\frac{1}{-2\pi i f}\left[e^{-2\pi i f T} - 1\right] = \frac{1}{T}\frac{e^{-\pi i f T}}{\pi f}\frac{-e^{-\pi i f T} + e^{\pi i f T}}{2i} = e^{-\pi i f T}\frac{\sin(\pi f T)}{\pi f T}$

$P_{output} = P_s \cdot |H(f)|^2 = \left[\frac{2}{1+(2\pi f)^2} + 0.1\right] \cdot \frac{\sin^2(\pi f T)}{(\pi f T)^2}$

d. Wiener filter $H_w(f) = \frac{P_x}{P_x + P_y} = \frac{\frac{2}{1+(2\pi f)^2}}{\frac{2}{1+(2\pi f)^2} + 0.1} = \frac{2}{2.1 + 0.4\pi^2 f^2}$



$H_w(0) \simeq 1$ $H_w(f) \simeq \frac{1}{2}$ if $0.4\pi^2 f^2 \simeq 2$ $\rightarrow f_{\frac{1}{2}} \simeq 0.7$



$|H| = \left|\frac{\sin(\pi f T)}{\pi f T}\right|$ $H(0) = 1$ $H(\frac{1}{T}) = 0$ $H(\frac{1}{2T}) \simeq 0.5$ (roughly)

$\Rightarrow$ choose $\frac{1}{2T} \simeq 0.7$ $\rightarrow T \simeq 0.7$ only approximate